AN ARITHMETICAL DUAL OF KUMMER'S QUARTIC SURFACE.*

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1. We shall use the well-established notations $\alpha \supset \beta$ to signify that α implies β, and $\alpha \equiv \beta$ to express the formal equivalence of α and β, viz. $\alpha \supset \beta$ *and* $\beta \supset \alpha$. Consider the homogeneous algebraic equation

$$\text{(A)} \qquad \Phi(x, y, z, w) = 0.$$

When x, y, z, w are the homogeneous coordinates of a point, (A) is the equation S' of a surface S. If x, y, z, w are assigned any other interpretation I, (A) is then called the equation E' of the equivalent E of S with respect to I. If $S' \equiv E'$, we shall call S, E duals of each other with respect to I; and if I is arithmetical, E is defined to be an arithmetical dual of S.

Let E be an arithmetical dual of S, and suppose that all the properties of integers implicit in E are transposed into a set P of properties of point configurations lying in a lattice space of the appropriate number of dimensions.† If now $P \supset E'$, then E, S being arithmetical duals, $P \supset S'$; and conversely $S' \supset P$. Hence since $P \equiv S'$ we may regard S in continuous point space as the equivalent or image of P' in the discrete lattice space. Another image of S will be glanced at in § 10. Clearly all of these considerations can be readily modified to hold for any system of equations representing loci in any space.

2. The most interesting of these duals appear to be those of the non-rational plane algebraic curves and the hyperelliptic r-folds in space of $r(r+1)/2$ dimensions. Obvious geometrical properties suggest analytical relations which dualize into interesting properties of numbers having many applications to the arithmetic of systems of higher forms. Thus we have a new application of geometry to numbers.

Here we briefly consider an arithmetical dual of the quartic surface with sixteen nodes, carrying the development up to the point of providing an immediate means for translating all the geometry of the surface directly

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† The E of Kummer's S considered in this paper can be interpreted in terms of lattice configurations lying upon two concentric spheres in a space of sixteen dimensions, but this is not developed here. This interpretation will be sufficiently evident from § 7 on applying the isomorphism there established to the rationalized form of the equation of S as given in Borchardt, *Crelle*, 83 (1877), p. 238; or Krause, *Die Transf. Hyperell. Funkt. erster Ord.* (Leipsig, 1886), p. 39; or Hudson, *Kummer's Quartic Surface* (Cambridge, 1905), p. 81. Rohn's parametric representations of S, *Math. Annalen*, 15 (1874), p. 315, based upon the transformation of the second order, give two more arithmetical duals distinct from E.

into arithmetic, the translation being reversible, so that from it the geometry can be recovered. Thus, for example, the 16_6 configuration of nodes and tropes is equivalent to a remarkable interlacing of properties of sets of three integers constructed from the decompositions of a pair of arbitrary integers into two sums of four squares, and conversely these imply the existence of the configuration.

3. Henceforth all letters l, m, n denote integers $\gtreqless 0$ unless further specified; *the l's are always even and the m's odd.* Write*

$$[\sum_{j=1}^{r} n_j] \equiv [n_1, n_2, \cdots, n_r],$$

and let either of these symbols denote $1, i, -1, -i$ $(i \equiv \sqrt{-1})$ according as

$$\sum_{j=1}^{r} n_j \equiv 0, 1, 2, 3 \bmod 4.$$

From the definition we have

$$[n_1, n_2, \cdots, n_r] = \prod_{j=1}^{r} [n_j]; \qquad [l] = (-1)^{l/2}; \qquad [m] = i(-1)^{(m-1)/2},$$

so that the symbol represents a real or imaginary unit according as the number of odd n_j is even or odd. We define $F(x, y, z)$ to be an arithmetical function if it exists and is single valued for x, y, z simultaneously integers $\gtreqless 0$, and otherwise is wholly arbitrary. If further conditions are imposed upon F it is called restricted; and unless the contrary is expressly noted, throughout the paper $F(x, y, z)$ denotes an unrestricted arithmetical function. An immediate consequence of this definition is of importance later (§ 10).

If

$$\sum_{j=1}^{r} F(n_{1j}, n_{2j}, n_{3j}) = \sum_{k=1}^{s} F(n_{1k}', n_{2k}', n_{3k}'),$$

then $r = s$, and in some order the triads (n_{1j}, n_{2j}, n_{3j}) are identical with the triads $(n_{1k}', n_{2k}', n_{3k}')$, two such being identical when and only when* $n_{ij} = n_{ik}'$ $(i = 1, 2, 3)$.

4. Let g_1, g_2, h_1, h_2 denote constant integers $\geqq 0$, and write $2n_j + g_j \equiv \nu_j$ $(j = 1, 2)$. Then the *general f-function* is defined by

$$f(\nu_1, \nu_2) \equiv f\begin{bmatrix} g_1 g_2 \\ h_1 h_2 \end{bmatrix}^{(n_1, n_2)} = [h_1\nu_1, h_2\nu_2] F(\nu_1, \nu_2, \nu_1\nu_2).$$

* The $\equiv$ here means algebraic identity. There can be no confusion between this and the sign of formal equivalence, as in each case the context indicates which is meant. The [ ] notation is merely to avoid the printing of complicated exponents.

† If a formal proof of this proposition is desired, it can easily be constructed from section II of "Arithmetical Paraphrases," to appear shortly in the *Trans. Am. Math. Soc.*

The symbol $\begin{bmatrix} g_1 g_2 \\ h_1 h_2 \end{bmatrix}$ is the *mark* of this f, and is even or odd according as $g_1 h_1 + g_2 h_2$ is even or odd. According as its mark is even or odd, f is *quasi-even* or *quasi-odd*. If the f just written is quasi-even, the function

$$\varphi(\nu_1, \nu_2) \equiv \varphi \begin{bmatrix} g_1 g_2 \\ h_1 h_2 \end{bmatrix}^{(n_1, n_2)} = [h_1\nu_1, h_2\nu_2] F(0, 0, \nu_1\nu_2)$$

is the *quasi-constant* corresponding to f. The integers ν_1, ν_2 are the *parameters of f or φ.*

For $g_1, g_2, h_1, h_2 = 0, 1$ there are thus 16 marks, of which 10 are even and 6 odd. Corresponding to these marks are 16 distinct f-functions which fall into sets of 4 each according to the residues mod 2 of the variables ν_1, ν_2:

$$(\nu_1, \nu_2) = (l_1, l_2),\ (m_1, l_2),\ (l_1, m_2),\ (m_1, m_2).$$

Transposing the entire theory of double theta marks to the present subject, we name each reduced mark, excluding (00), by the particular dyad of odd marks to which it is congruent.* If now (ij) is any one of the 16 reduced marks, (00) included, the $f(\nu_1, \nu_2)$ whose mark is (ij) is denoted by $f_{ij}(\nu_1, \nu_2)$, or simply by f_{ij}; and if (ij) is even, the corresponding quasi-constant is $\varphi_{ij}(\nu_1, \nu_2)$ or φ_{ij}. In this notation the order of i, j in the suffix is indifferent. We thus have the following system of 16 f-functions in which the parity of the variables is indicated by the l, m notation (§ 3):

$$\begin{aligned}
f_{00} &= F(l_1, l_2, l_1 l_2), & f_{23} &= [l_1] f_{00}, & f_{45} &= [l_2] f_{00}, & f_{16} &= [l_1, l_2] f_{00};\\
f_{12} &= F(m_1, l_2, m_1 l_2), & f_{13} &= [m_1] f_{12}, & f_{36} &= [l_2] f_{12}, & f_{26} &= [m_1, l_2] f_{12};\\
f_{56} &= F(l_1, m_2, l_1 m_2), & f_{14} &= [l_1] f_{56}, & f_{46} &= [m_2] f_{56}, & f_{15} &= [l_1, m_2] f_{56};\\
f_{34} &= F(m_1, m_2, m_1 m_2), & f_{24} &= [m_1] f_{34}, & f_{35} &= [m_2] f_{34}, & f_{25} &= [m_1, m_2] f_{34}.
\end{aligned}$$

The six quasi-odd functions are

$$f_{13}, f_{24}, f_{46}, f_{35}, f_{26}, f_{15};$$

and the ten quasi-constants are

$$\varphi_{00}, \varphi_{12}, \varphi_{56}, \varphi_{34}, \varphi_{23}, \varphi_{14}, \varphi_{45}, \varphi_{36}, \varphi_{16}, \varphi_{25}.$$

Henceforth in all functions of f's and φ's, the φ's are regarded as being of degree zero, so that *in determining the degree of any expression involving f's and φ's the φ's are to be considered as absolute constants.* Thus $\varphi_{13}^2 f_{24}^2 + \varphi_{35}^2 f_{15}^2 = 0$ is a homogeneous linear relation between squares of f-functions.

5. The arithmetic-geometric translations mentioned in § 2 depend upon

* See Harkness and Morley, "Treatise on the Theory of Functions," ch. 8, whose notation we shall follow throughout; also Weber, "Über die Kummersche Fläche," u.s.w., *Crelle*, 84 (1878), especially pp. 332–334.

a form of symbolic product, or henceforth simply product, of $(r - s)$ f-functions and s quasi-constants, $0 \leqq s \leqq r$. These products are taken with respect to a pair of integers $n_1, n_2 > 0$ of preassigned linear forms mod 4:

$$n_j \equiv t_j \bmod 4; \ t_j \text{ constant}, \quad (j = 1, 2).$$

Let all the g, h denote constants, each of which is one of the numbers 0, 1. Write

$$\begin{bmatrix} g_{1i}, & g_{2i} \\ h_{1i}, & h_{2i} \end{bmatrix} \equiv (i), \qquad 2n_{ji} + g_{ji} \equiv \nu_{ji} \quad (j = 1, 2),$$

and suppose that precisely t_1 of the g_{1i} each $= 1$, so that precisely t_1 of the ν_{1i} are odd, and similarly for t_2 and the g_{2i}, ν_{2i}. The f's or φ's for the marks (i), (k) are

$$f_i(\nu_{1i}, \nu_{2i}) \equiv f_i = [h_{1i}\,\nu_{1i}, h_{2i}\nu_{2i}]F(\nu_{1i}, \nu_{2i}, \nu_{1i}\nu_{2i}),$$
$$\varphi_k(\nu_{1k}, \nu_{2k}) \equiv \varphi_k = [h_{1k}\nu_{1k}, h_{2k}\nu_{2k}]F(0, 0, \nu_{1k}\nu_{2k}),$$
$$(i = s + 1, s + 2, \cdots, r;\ k = 1, 2, \cdots, s).$$

Put

$$h \equiv \sum_{i=1}^{r} (h_{1i}\nu_{1i} + h_{2i}\nu_{2i}), \quad \xi_{12} \equiv \sum_{i=1}^{r} \nu_{1i}\nu_{2i}, \quad \xi_j \equiv \sum_{j=s+1}^{r} \nu_{ji} \quad (j = 1, 2).$$

Then, the Σ extending to all $\nu_{ji} \gtreqless 0$ such that

$$n_j = \nu_{j1}^2 + \nu_{j2}^2 + \cdots + \nu_{jr}^2 \quad (j = 1, 2),$$

the *product with respect to* n_1, n_2 of the s quasi-constants and $(r - s)$ f-functions just written is defined by

$$\prod_{i=1}^{s}{}' \varphi_i(\nu_{1i}, \nu_{2i}) \cdot \prod_{i=s+1}^{r}{}' f_i(\nu_{1i}, \nu_{2i}) \equiv \Sigma[h]F(\xi_1, \xi_2, \xi_{12}),$$

and the *type** of this product is $\{t_1, t_2\}, \equiv \{t_2, t_1\}$. The accent in Π' indicates that the multiplication is purely symbolic.

Note that the sum on the right consists of only a finite number of terms. For if $N_r(n, s)$ is the total number of representations of n as a sum of r squares precisely s of which are odd and occupy fixed positions, the number of terms is $N_r(n_1, t_1)N_r(n_2, t_2)$. Note also (cf. § 3) that if $t_1 + t_2$ is even, $[h]$ is real. In the products connected with Kummer's surface it will be seen at once that each f_i, φ_k occurs an even number of times. Hence in these products $[h]$ is always real, and we need not again refer to this.

When a_j of the marks (1), (2), $\cdots$, (s) each $= j$, and b_k of the marks $(s + 1)$, $(s + 2)$, $\cdots$, (r) each $= k$, the product is written $\varphi_1^{a_1}\varphi_2^{a_2} \cdots f_1^{b_1}f_2^{b_2} \cdots$; and clearly the order of the factors $\varphi_1^{a_1}, \cdots, f_1^{b_1}, \cdots$ in this is im-

* The type is not important for this paper, and is included here merely for completeness. It plays an essential part in the detailed discussion of the arithmetical nodes and tropes on E, cf. § 9, second footnote.

material. Hence such multiplication of φ's and f's is commutative. We shall not be concerned here with its associative and distributive aspects. By definition $\varphi_i^0, f_i^0 = 1$; and unity in this multiplication is to have all the formal properties of unity in ordinary multiplication, so that unit factors may be suppressed.

The *parameters of the product* are n_1, n_2. These ultimately play a part, in one geometrical interpretation of the arithmetic, analogous to that of v_1, v_2, the parameters of a point on the Kummer surface; cf. § 9. Any product p_i being a sum with respect to given parameters, the meaning of $\Sigma_i a_i p_i$ where the a_i are absolute constants is evident. Note that

$$a_i p_i + a_j p_i = (a_i + a_j) p_i$$

when and only when both p_i are with respect to the same parameters. To indicate that each p_i in the sum is with respect to n_1, n_2 we write $n_1, n_2 | \Sigma_i a_i p_i$. If now

(1) $$n_1, n_2 | \sum_i a_i p_i = 0,$$

it follows at once from the definitions that we may replace n_j by n_j' where $n_j' \equiv t_j \bmod 4$ $(j = 1, 2)$. For each pair (n_1', n_2'), (1) gives a relation

(2) $$n_1', n_2' | \sum_i a_i p_i = 0$$

between products. To signify that we are considering the totality of relations of the form (2) for all (n_1', n_2'), we shall write (1) without the $n_1, n_2 |$, thus,

(3) $$\Sigma a_i p_i = 0.$$

It is important to note that (3) is of the form $P(f, \varphi) = 0$, where $P(f, \varphi)$ is a polynomial in f's and φ's.

Consider now $R(f, \varphi) = 0$ where $R(f, \varphi)$ is a rational algebraic function of f's and φ's. As yet this relation has no significance. By purely formal algebraic reductions $R(f, \varphi) = 0$ may be written $P(f, \varphi) = 0$, $P(f, \varphi)$ as above, and we define this to be the meaning of $R(f, \varphi) = 0$, again emphasizing the significance of the omission of $n_1, n_2 |$. Thus in all that follows, $R(f, \varphi) = 0$ and $P(f, \varphi) = 0$, its polynomial equivalent, are regarded as identical.

6. Denote by $R(\theta, c) = 0$ a rational algebraic relation with rational coefficients $\gtreqless 0$ between the sixteen double theta functions θ_{ij} and the related constants* c_{ij}.

* The notation is that of Harkness and Morley, loc. cit. The restriction that R be rational is merely to shorten the following proof. By a few simple changes the theorem may be restated for R algebraic with algebraic number coefficients, but this is not required for the dual of Kummer's surface.

By algebraic reductions $R(\theta, c) = 0$ may be cast in the form $P(\theta, c) = 0$, where now P is a polynomial with integral coefficients $\gtreqless 0$. We shall regard $R(\theta, c) = 0$, $P(\theta, c) = 0$ as identical statements. In these replace θ_{ij}, c_{ij} by f_{ij}, φ_{ij} respectively, and denote the results by $R(f, \varphi) = 0$, $P(f, \varphi) = 0$. It is not immediately evident, of course, that $P(f, \varphi) = 0$, or its formal equivalent $R(f, \varphi) = 0$, is true. The fundamental theorem is:

$$\{P(\theta, c) = 0\} \equiv \{P(f, \varphi) = 0\},$$

(the sign $\equiv$ being, as in the next section, that defined in § 1), or what is the same thing,

$$\{R(\theta, c) = 0\} \equiv \{R(f, \varphi) = 0\}.$$

No doubt it is easy to prove this from first principles. It is less tedious, however, to proceed as follows.

7. Clearly $P(f, \varphi) = 0$ is a finite sum relation of the form

$$\sum_k a_k F(\alpha_k, \beta_k, \gamma_k) = 0,$$

in which $a_k, \alpha_k, \beta_k, \gamma_k$ are integers, and it is easily seen that if $g(x, y, z)$ is any restricted arithmetical function such that $g(x, y, z) = -g(-x, -y, -z)$, then $\Sigma_{\pm} g(\pm x, \pm y, \pm x \times \pm y)$ is not identically zero. In each f, φ replace $F(\alpha_k, \beta_k, \gamma_k)$ by $\sin(\alpha_k x + \beta_k y + \gamma_k z)$, where x, y, z are parameters, and denote by $P_1(f, \varphi) = 0$ what $P(f, \varphi) = 0$ becomes under this substitution; and similarly for $F(\alpha_k, \beta_k, \gamma_k)$ replaced by $\cos(\alpha_k x + \beta_k y + \gamma_k z)$, giving $P_2(f, \varphi) = 0$. Write $P_3(f, \varphi) = iP_1(f, \varphi) + P_2(f, \varphi)$, $(i = \sqrt{-1})$.

We shall be concerned with the implications of the five propositions defined by

$\alpha \equiv \{P(f, \varphi) = 0\}$, $\beta \equiv \{R(\theta, c) = 0\}$, $\alpha_j \equiv \{P_j(f, \varphi) = 0\}$, $(j = 1, 2, 3)$, and will show that

$$(4)\quad \alpha \supset \alpha_3, \qquad (6)\quad \alpha_3 \supset \beta,$$
$$(5)\quad \beta \supset \alpha_3, \qquad (7)\quad \alpha_3 \supset \alpha;$$

whence from (4), (6) it will follow that $\alpha \supset \beta$, and from (5), (7) that $\beta \supset \alpha$, and therefore from these $\alpha \equiv \beta$, which is the theorem.

From the definition of F (§ 3), (4) is obvious. To prove (5) and (6) we observe that $P_3(f, \varphi)$ is the coefficient of $p_1^{n_1}p_2^{n_2}$ in $R(\theta, c) = 0$ when the expansion of the general theta function

$$\theta \begin{bmatrix} g_1 g_2 \\ h_1 h_2 \end{bmatrix}^{(x/\pi,\, y/\pi,\, 4\tau_{11},\, 2z/\pi,\, 4\tau_{22})}, \quad (z = \pi\tau_{12}/2),$$

is written in the form

$$\Sigma p_1^{\nu_1^2} p_2^{\nu_2^2} [h_1\nu_1, h_2\nu_2] \exp i(\nu_1 x + \nu_2 y + \nu_1\nu_2 z), \quad (p_j = \exp i\pi\tau_{jj}),$$

the Σ referring to all $\nu_j = 2n_j + g_j$ $(j = 1, 2)$.

To prove (7) consider two restricted arithmetical functions (cf. § 3) ψ_1, ψ_2, the only restrictions upon them being

$$\psi_j(u, v, w) = (-1)^j \psi_j(-u, -v, -w), \quad (j = 1, 2),$$

and consider the propositions α_j' defined by the formal equivalences

$$\alpha_j' \equiv \{\sum_k a_k \psi_j(\alpha_k, \beta_k, \gamma_k) = 0\}, \quad (j = 1, 2).$$

Then it may be shown without difficulty* that

$$(8) \qquad \alpha_j \supset \alpha_j', \quad (j = 1, 2).$$

Now choose for the ψ_j the restricted arithmetical functions defined by

$$2\psi_j(u, v, w) = F(u, v, w) + (-1)^j F(-u, -v, -w), \quad (j = 1, 2),$$

which obviously satisfy the restrictions imposed upon the ψ_j. Substitute these values of the ψ_j in the α_j' respectively, getting α_j''. Then from (8)

$$(9) \qquad \alpha_j \supset \alpha_j'', \quad (j = 1, 2).$$

If now $i\alpha_1$ denotes the result of replacing sin $(\alpha_k x + \beta_k y + \gamma_k z)$ in α_1 by i times itself, we have $i\alpha_1 \supset \alpha_1$, and hence from (9)

$$(10) \qquad i\alpha_1 \supset \alpha_1'' \quad \textit{and} \quad \alpha_2 \supset \alpha_2''.$$

But obviously

$$(\alpha_1'' \textit{ and } \alpha_2'') \supset \alpha, \text{ also } (i\alpha_1 \textit{ and } \alpha_2) \equiv \alpha_3;$$

while from (10),

$$(i\alpha_1 \textit{ and } \alpha_2) \supset (\alpha_1'' \textit{ and } \alpha_2'').$$

Hence $\alpha_3 \supset \alpha$, which is (7).

8. The theorem of § 6 enables us to transpose the entire theory of the rational algebraic relations between the θ_{ij}, c_{ij} to the present subject. In particular there is here a system of 16 Rosenhain hexads of which one is†

$$f_{13}, f_{35}, f_{15}, f_{24}, f_{46}, f_{26}$$

such that there is a linear relation (cf. § 4 end) between the squares of any four f-functions belonging to the same hexad; also the sixteen f_{ij} fall into 60 Göpel tetrads, the four f-functions in a tetrad being connected by a homogeneous biquadratic relation. Again, between the squares of any five f-functions, no four of which belong to a hexad, there is a linear relation; and we may state the general theorem* that the squares of four f-functions are linearly related when and only when their marks belong to a Rosenhain

* This is included as a very special case of a theorem proved in "Arithmetical Paraphrases," Part I, Sec. II, cited in § 3, footnote.

† The complete system may be written down from the table in Harkness and Morley, loc. cit., pp. 353–354.

‡ Harkness and Morley, loc. cit., p. 368.

hexad, and the squares of any five f-functions no four of which are in one hexad are linearly related. All of these results have immediate arithmetical interpretations in terms of representations of n_1, n_2 either as sums of four squares (for the quadratic relations), or as sums of sixteen squares (for the biquadratic). The subject being extensive we shall not go into it here.

By the theorem of § 6, the Göpel relations for theta functions (and constants) are formally equivalent to the same for f-functions (and quasi-constants), so that the Kummer surface gives in this way 60 arithmetical duals. We shall sketch a means for translating the arithmetic into terms of configurations of lattice points in S_r (space of r dimensions), a lattice point being one all of whose coördinates are integers. Finally it will be shown that the lattice properties in S_r can be mapped onto a point configuration in S_3. In each case the configuration is formally equivalent to the arithmetical dual, which in turn is formally equivalent to the equation of the surface, so that the existence of the point configuration implies that of the surface, and conversely.

9. Let ν_a, ν_a' be the parameters (§ 3) of the f whose mark is a, and consider the tetrad $T \equiv (f_a, f_b, f_c, f_d)$. As the eight parameters $\nu_a, \nu_a', \cdots, \nu_d, \nu_d'$ take all their possible values, we shall say that T generates a spread. The totality of all spreads obtained by choosing a, b, c, d and F (cf. § 3) in all possible ways is called f-space, and each T for constant values of the eight parameters involved is a point in f-space (or the coördinates of a point in f-space). The analogously defined $(\varphi_a, \varphi_b, \varphi_c, \varphi_d)$, in which the φ's are quasi-constants, is a singular point of f-space. The coördinates of a general point in f-space are (f_a, f_b, f_c, f_d), in which all the parameters are general. In the same way we define the general f-line $(f_a, f_b, f_c, f_d, f_e, f_g)$, $a, \cdots, g$ being any six marks, and similarly a singular f-line on replacing f's by φ's when all the marks are even. We remark that, the p's being products, (p_1, p_2, p_3, p_4) is clearly an f-point; and likewise for hexads of products and f-lines. The analogies between f-points and lines and the points and lines of S_3 are obvious, and need not be further elaborated.

Suppose now that between the coördinates $(f_1, f_2, f_3, f_4) \equiv (x, y, z, w)$ of an f-point there is a relation of the form (A), § 1. Remembering that by § 4 (end) quasi-constants are to be considered as being of degree zero, and that by § 5(3) each product in (A) is with respect to the same parameters n_1, n_2 which are general, we shall call (A) the equation of the surface E (§ 1), in f-space, $x = f_1$, $y = f_2$, $z = f_3$, $w = f_4$ the parametric equations of E, and n_1, n_2 the parameters of a point on E. Replacing f-functions throughout in the above by quasi-constants, we similarly define singular surfaces in f-space. The singular surfaces have no immediate geometrical

analogues; their interest is arithmetical, and need not be considered here.* In the same way is defined the line-equation of an f-surface or of a singular f-surface. As one parameter varies, the other being constant, the associated point on the surface traces out what may be called an f-curve; and the analogy with geometry may obviously be continued step by step. Thus, to define a node on the f-surface E, we equate to zero the four formal partial derivatives with respect to the f-coördinates. It does not follow, of course, that any one of the four new relations thus derived between products is true. But if it be possible in the four derived relations so to replace the variables (f-functions) x, y, z, w by quasi-constants $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ respectively that the relations become simultaneously true, $(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ is defined to be a node on E. It follows from § 7 that the E of Kummer's S has sixteen nodes, and it is sufficiently evident that the geometry of nodes on S can be transferred to a 'geometry' of 'nodes' on E, and conversely.† All of this geometry on E has a simple interpretation in S_3 which we shall briefly consider next, showing how the translation is effected in a general case. It is not the most general case, for we have excluded the possibility of imaginary $[h]$, but the perfectly general case presents no difficulty, and is treated in essentially the same way. In the case considered we end with the identity of two point configurations in a lattice space of three dimensions; in the most general case the final result is that two pairs of point configurations are thus severally identical. The case treated appears to be the more important. It includes the E of Kummer's S.

10. Consider three fixed points U, V_1, V_2 and two concentric spheres σ_1, σ_2, centers at the origin of the respective radii a_1, a_2 in S_r (§ 9), and let T_j denote the tangent plane (tangent S_{r-1}) at the point P_j on σ_j $(j = 1, 2)$. Let δ_j, δ_{12}, δ_{21} denote the respective perpendicular distances from U to T_j, from P_1 to T_2 and from P_2 to T_1. It is easily seen that

$$a_1\delta_{21} + a_1^2 = a_2\delta_{12} + a_2^2.$$

With reference to rectangular axes in S_3 the point P_{12} whose coördinates

* Interpreted similarly to the non-singular surfaces in § 10 (end), they give point configurations lying in one plane.

† The equivalent arithmetic is obtained in an obvious manner. For example the the theorem that a particular set of six nodes is in one plane is formally equivalent to a theorem that six products (§ 5), in which $r = s = 2$, are linearly related. The six products are sums of terms of the form $[h]F(\xi_1, \xi_2, \xi_{12})$, where the ξ's refer in this example to all the representations of the parameters n_1, n_2 in two sums of four squares, either 2 or 4 of which are odd and the rest even. The number of odd squares, also the $[h]$, are not the same for all sets of six coplanar nodes. Finally, as in the equivalent theta theory, all these relations between (symbolic) products can be summed up in the statement that a certain matrix is orthogonal, cf. Hudson, loc. cit., chs. III, XVI.

are either of the identical triads

$$(a_1\delta_1 + a_1^2,\ a_2\delta_2 + a_2^2,\ a_1\delta_{21} + a_1^2), \qquad (a_1\delta_1 + a_1^2,\ a_2\delta_2 + a_2^2,\ a_2\delta_{12} + a_2^2),$$

is called the image through U of the pair P_1, P_2, or when U is understood, simply the image of P_1, P_2.

Let δ_j' denote the perpendicular distance of V_j to T_j. Then it is readily seen that if the squares of the radii a_1, a_2 are integers, and V_j, P_j lattice points, h_j, defined by

$$h_j = a_j\delta_j' + a_j^2, \qquad h \equiv h_1 + h_2, \qquad (j = 1, 2),$$

are integers. We shall call h_j the index of P_j, and h the index of P_{12}. *Henceforth P_1, P_2 are lattice points.* Images P_{12} are now segregated into four classes K_j $(j = 0, 1, 2, 3)$, all images in K_j having their indices $\equiv j \bmod 4$. We shall consider henceforth only K_0 and K_2. A configuration C_0 of images in S_3 is called even when all its images belong to K_0; if all the images of C_2 belong to K_2, C_2 is odd. Several even C_0's together are regarded as forming a single C_0; likewise for C_2's, so that the C_j is the logical sum of all the images in the several C_j', C_j'', $\cdots$,

$$C_j = C_j' + C_j'' + \cdots \quad (j = 1, 2).$$

If in C_j a particular image P_{12} occurs precisely k times, P_{12} is *multiple of order k* in C_j; and if all the images of C_0 coincide with all those of C_2, C_0 and C_2 are identical when and only when images occupying the same positions in both are of equal multiplicities.

Returning to § 5 we shall consider only the case in which h there defined is even, so that $[h]$ is real. In the notation of § 5 choose for the radii of σ_1, σ_2 of this section, $\sqrt{n_1}$, $\sqrt{n_2}$, let the origin be the common center, and take for U, V_j, P_j the points

$$\begin{aligned} U &\equiv (0, 0, \cdots, 0, 1, 1, \cdots, 1), \quad (s \text{ zeros}, r - s \text{ units}), \\ V_j &\equiv (h_{j1}, h_{j2}, \cdots, h_{jr}), \quad (j = 1, 2), \\ P_j &\equiv (\nu_{j1}, \nu_{j2}, \cdots, \nu_{jr}), \quad (j = 1, 2). \end{aligned}$$

Assume $r \geqq 4$ (the only cases of importance), so that, any integer > 0 being in several ways a sum of four integral squares, σ_j always passes through lattice points P_j. For the U, V_j, P_j as just defined, it is easily seen that P_{12}, the image through U of P_1, P_2, is, in the notation of § 5, (ξ_1, ξ_2, ξ_{12}), and that h as there defined is its index. Hence to each term $[h](\xi_1, \xi_2, \xi_{12})$ of the product in § 5 corresponds an image of odd or even index, and to all the terms in the product correspond an even and an odd configuration.

Suppose now that we have a homogeneous algebraic relation $H = 0$

of degree $(r - s)$ (cf. § 4 end) between k such products. The k even configurations corresponding respectively to the k products form a single C_0; likewise the k odd configurations form a single C_2, and by § 3 (end) we see at once that $H = 0$ is formally equivalent to the statement that these single odd and even configurations are identical.

It should be of interest, for physical reasons, to find the corresponding discrete image of the special case of Kummer's surface known as the wave surface. The arithmetic for this case appears to be less elegant than that for the general surface.